

Black-Scholes formula - a Heston approach

Bogdan Patrut

Vasile Alecsandri University of Bacau, Faculty of Sciences, Romania
bogdan@edusoft.ro

Tiberiu Socaciu

Stefan cel Mare University of Suceava, Faculty of Economics and Public Administration, Romania
tibisocaciu@yahoo.com

Abstract

In this paper we will compare Black-Scholes formula with a particular case of Heston formula, both solutions of the same problem.

Keywords: Black-Scholes model, Heston model, comparing analytic solutions.

1. Introduction

As an extension of Black and Scholes (1973) model:

$$dS = \mu S dt + \sigma^{0.5} S dW, \quad (1)$$

where

- a) W is an Wiener process;
- b) μ is a constant named drift;
- c) σ is a constant named volatility;
- d) S is a process for a traded asset.

Steven and Heston (1993) define a new model with a stochastic volatility, see equation (2):

$$\begin{aligned} dS &= \mu S dt + v^{0.5} S dW \\ dv &= \theta (\sigma - v) dt + \xi v^{0.5} dB, \end{aligned} \quad (2)$$

where:

- a) ω is long term of volatility;
- b) θ is return factor to mean of volatility (σ);
- c) ξ is volatility of volatility;
- d) B and W are Wiener standard processes ρ -correlated;
- e) S is a stochastic process for a traded asset;
- f) v is a stochastic process for volatility.

This model was extended by Christoffersen, Heston and Jacobs (2009) as a model with two stochastic semi-volatilities. In our opinion, this model can be generalized as a stochastic model with q ($q > 0$) stochastic partial-(or semi-)volatilities.

2. Solutions of the models in mirror

SDE for BS model	$dS = \mu S dt + \sigma^{0.5} S dW$
Analytic solutions for european calls	$V(s,t) = N(d_1) S - N(d_2) E \exp(-r(T-t))$ $d_1 = \sigma^{-1} (T-t)^{-0.5} [\ln(S/E) + (r + \frac{1}{2} \sigma^2)(T-t)]$ $d_2 = \sigma^{-1} (T-t)^{-0.5} [\ln(S/E) + (r - \frac{1}{2} \sigma^2)(T-t)]$

with E strike with BS model	$N(x) - pdf \text{ of } N(0,1) \text{ distribution}$
Official references for model and analytic solutions for BS model	(Black, F. & Scholes, M., 1973)
SDEs for H model	$dS = \mu S dt + v^{0.5} S dW$ $dv = \theta (\sigma - v) dt + \xi v^{0.5} dB$
Analytic solutions for european calls with E strike with H model	$V(s,v,t) = S P_1 - E \exp(-r(T-t)) P_2$ $P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[\frac{\exp(i\phi \log(E)) f_j(x,v,t,\phi)}{i\phi} \right] d\phi$ $f_j(x, v, t, z) = \exp(C_j(T-t, z) + D_j(T-t, z) z + i z x)$ $C_j(t, z) = r z i t + a [(b_j - \rho \xi z i + d_j) t - 2 \log(1 - g_j \exp(d_j r)) + 2 \log(1 - g_j)] \xi^2$ $D_j(t, z) = [(b_j - \rho \xi z i + d_j) [1 - \exp(d_j r)]] \xi^2 [1 - \exp(d_j r)]^{-1}$ $g_j = [b_j - \rho \xi z i + d_j] [b_j - \rho \xi z i - d_j]^{-1}$ $d_j = [(b_j - \rho \xi z i)^2 - \xi^2 (2 u_j z i - z^2)]^{0.5}$ $u_1 = 1/2$ $u_2 = -1/2$ $a = k \theta$ $b_1 = k + \lambda - \rho \xi$ $b_2 = k + \lambda$ $j=1,2$
Official references for model and analytic solutions for H model	(Steven & Heston, 1993)

3. Links between solutions?

We can point that Heston model is a generalization of Black-Scholes model. For

$$v = \sigma \quad (3)$$

id est:

$$dv = 0 \quad (4)$$

the two models are identical.

But

$$dv = 0 \quad (5)$$

is same with

$$\theta = \xi = 0, \quad (6)$$

that means:

$$a = k \theta = 0, \quad (7)$$

$$b_1 = k + \lambda - \rho \xi = k + \lambda = b_2, \quad (8)$$

$$d_j = [(b_j - \rho \xi z i)^2 - \xi^2 (2 u_j z i - z^2)]^{0.5} = [(b_j - \rho \xi z i)^2]^{0.5} = |b_j|, \quad (9)$$

$$g_j = [b_j - \rho \xi z i + d_j] [b_j - \rho \xi z i - d_j]^{-1} = [b_j + |b_j|] [b_j - |b_j|]^{-1} = 0, \quad (10)$$

if assume that

$$k + \lambda < 0$$

$$D_j(t, z) = [(b_j - \rho \xi z i + d_j) [1 - \exp(d_j r)]] \xi^2 [1 - \exp(d_j r)]^{-1} = [b_j + |b_j|] [1 - \exp(|b_j| r)] \xi^2 [1 - \exp(|b_j| r)]^{-1} = 0, \quad (11)$$

if assume that

$$[b_j + |b_j|] \xi^2 = 0/0 = 0, \quad (12)$$

$$C_j(t, z) = r z i t + a [(b_j - \rho \xi z i + d_j) t - 2 \log(1 - g_j \exp(d_j r)) + 2 \log(1 - g_j)] \xi^2 = r z i t, \quad (13)$$

if assume that

$$+ a \xi^2 = 0/0 = 0. \quad (14)$$

$$f_j(x, v, t, z) = \exp(C_j(T - t, z) + D_j(T - t, z) z + i z x) = \exp(r z i t + i z x)$$

$$\begin{aligned} P_j &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[\frac{\exp(i \varphi \log(E)) f_j(x, v, t, \varphi)}{i \varphi} \right] d\varphi = \frac{1}{2} + \int_0^\infty \operatorname{Re} [\exp(i z \log(E)) \exp(r z i t \\ &+ i z x) i^{-1} z^{-1}] dz = \frac{1}{2} + \int_0^\infty \operatorname{Re} [[\cos(z \log(E)) + i \sin(z \log(E))] [\cos(r z t + z x) \\ &+ i \sin(r z t + z x)] i^{-1} z^{-1}] dz = \frac{1}{2} + \int_0^\infty \cos(z \log(E)) \sin(r z t + z x) + \sin(z \log(E)) \cos(r z t + z x)] \\ &z^{-1}] dz = \frac{1}{2} + \int_0^\infty [\sin(z \log(E) + r z t + z x)] z^{-1}] dz. \quad (15) \end{aligned}$$

4. Comments and further works

We expected that the two solutions are identical. Because not getting the same result on the two different routes, results that Heston solution has a little inconsistency on some particular cases, like $\xi = 0$ (we use that $0/\xi = 0!$). Therefore, a revision of the solution Heston by treating individual cases. We intend to do so in the future.

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