Black-Scholes formula - a Heston approach

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Abstract
In this paper we will compare Black-Scholes formula with a particular case of Heston formula, both solutions of the same problem.
Keywords: Black-Scholes model, Heston model, comparing analytic solutions.

1. Introduction
As an extension of Black and Scholes (1973) model:

$$dS = \mu S\, dt + \sigma^{0,5} S\, dW, \quad (1)$$

where
a) W is an Wiener process;
b) μ is a constant named drift;
c) σ is a constant named volatility;
d) S is a process for a traded asset.

Steven and Heston (1993) define a new model with a stochastic volatility, see equation (2):

$$dS = \mu S\, dt + v^{0,5} S\, dW$$
$$dv = \theta\,(\sigma - v)\, dt + \xi v^{0,5}\, dB, \quad (2)$$

where:
a) ω is long term of volatility;
b) θ is return factor to mean of volatility (σ);
c) ξ is volatility of volatility;
d) B and W are Wiener standard processes ρ-correlated;
e) S is a stochastic process for a traded asset;
f) v is a stochastic process for volatility.

This model was extended by Christoffersen, Heston and Jacobs (2009) as a model with two stochastic semi-volatilities. In our opinion, this model can be generalized as a stochastic model with q (q>0) stochastic partial-(or semi-)volatilities.

2. Solutions of tho models in mirror

SDE for BS model	$dS = \mu\, S\, dt + \sigma^{0,5}\, S\, dW$
Analytic solutions for european calls	$V(s,t) = N(d_1)\, S - N(d_2)\, E\, exp(-r\,(T-t))$ $d_1 = \sigma^{-1}\,(T-t)^{-0,5}\,[ln(S/E) + (r + ½\,\sigma^2)(T-t)]$ $d_2 = \sigma^{-1}\,(T-t)^{-0,5}\,[ln(S/E) + (r - ½\,\sigma^2)(T-t)]$

with E strike with BS model	$N(x)$ – *pdf of* $N(0,1)$ distribution
Official refferences for model and analytic solutions for BS model	(Black, F. & Scholes, M., 1973)
SDEs for H model	$dS = \mu S\, dt + v^{0,5} S\, dW$ $dv = \theta(\sigma - v)\, dt + \xi v^{0,5}\, dB$
Analytic solutions for european calls with E strike with H model	$V(s,v,t) = S P_1 - E \exp(-r(T-t)) P_2$ $P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp(i\varphi \log(E)) f_j(x,v,t,\varphi)}{i\varphi}\right] d\varphi$ $f_j(x, v, t, z) = \exp(C_j(T-t, z) + D_j(T-t, z)\, z + i\, z\, x)$ $C_j(t, z) = r\, z\, i\, t + a\, [(b_j - \rho\, \xi\, z\, i + d_j)\, t - 2\log(1 - g_j \exp(d_j\, r))$ $+ 2\log(1 - g_j)]\, \xi^{-2}$ $D_j(t, z) = [(b_j - \rho\, \xi\, z\, i + d_j]\, [1 - \exp(d_j\, r))]\, \xi^{-2}\, [1 - \exp(d_j\, r))]^{-1}$ $g_j = [b_j - \rho\, \xi\, z\, i + d_j]\, [b_j - \rho\, \xi\, z\, i - d_j]^{-1}$ $d_j = [(b_j - \rho\, \xi\, z\, i)^2 - \xi^2\, (2\, u_j\, z\, i - z^2)]^{0,5}$ $u_1 = ½$ $u_2 = -½$ $a = k\, \theta$ $b_1 = k + \lambda - \rho\, \xi$ $b_2 = k + \lambda$ $j=1,2$
Official references for model and analytic solutions for H model	(Steven & Heston, 1993)

3. Links beetween solutions?

We can point that Heston model is a generalization of Black-Scholes model. For

$$v = \sigma \quad (3)$$

id est:

$$dv = 0 \quad (4)$$

the two models are identical.

But

$$dv = 0 \quad (5)$$

is same with

$$\theta = \xi = 0, \quad (6)$$

that means:

$$a = k\,\theta = 0, \quad (7)$$

$$b_1 = k + \lambda - \rho\,\xi = k + \lambda = b_2\,, \quad (8)$$

$$d_j = [(b_j - \rho\,\xi\,z\,i)^2 - \xi^2\,(2\,u_j\,z\,i - z^2)]^{0,5} = [(b_j - \rho\,\xi\,z\,i)^2]^{0,5} = |b_j|, \quad (9)$$

$$g_j = [b_j - \rho\,\xi\,z\,i + d_j]\,[b_j - \rho\,\xi\,z\,i - d_j]^{-1} = [b_j + |b_j|]\,[b_j - |b_j|]^{-1} = 0, \quad (10)$$

if assume that

$$k + \lambda < 0$$

$$D_j(t, z) = [(b_j - \rho\,\xi\,z\,i + d_j]\,[1 - exp(d_j\,r))]\,\xi^2\,[1 - exp(d_j\,r))]^{-1} = [b_j + |b_j|]\,[1 - exp(|b_j|\,r))]\,\xi^2\,[1 - exp(|b_j|\,r))]^{-1} = 0, \quad (11)$$

if assume that

$$[b_j + |b_j|]\,\xi^{-2} = 0/0 = 0, \quad (12)$$

$$C_j(t, z) = r\,z\,i\,t + a\,[(b_j - \rho\,\xi\,z\,i + d_j)\,t - 2\,log(1 - g_j\,exp(d_j\,r)) + 2\,log(1 - g_j)]\,\xi^{-2} = r\,z\,i\,t, \quad (13)$$

if assume that

$$+\,a\,\xi^{-2} = 0/0 = 0. \quad (14)$$

$$f_j(x, v, t, z) = exp(C_j(T - t, z) + D_j(T - t, z)\,z + i\,z\,x) = exp(r\,z\,i\,t + i\,z\,x)$$

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp(i\varphi\log(E))\,f_j(x,v,t,\varphi)}{i\varphi}\right]d\varphi = ½ + \int_0^\infty Re\,[\,exp(i\,z\,log(E))\,exp(r\,z\,i\,t$$

$$+\,i\,z\,x)\,i^{-1}\,z^{-1}\,]\,dz = ½ + \int_0^\infty Re\,[\,[cos(z\,log(E)) + i\,sin(z\,log(E))]\,[cos(r\,z\,t + z\,x)$$

$$+\,i\,sin(r\,z\,t + z\,x)]\,i^{-1}\,z^{-1}\,]\,dz = ½ + \int_0^\infty cos(z\,log(E))\,sin(r\,z\,t + z\,x) + sin(z\,log(E))\,cos(r\,z\,t + z\,x)\,]$$

$$z^{-1}\,]\,dz = ½ + \int_0^\infty [\,sin(z\,log(E) + r\,z\,t + z\,x)\,]\,z^{-1}\,]\,dz. \quad (15)$$

4. Comments and further works

We expected that the two solutions are identical. Because not getting the same result on the two different routes, results that Heston solution has a little inconsistency on some particular cases, like $\xi = 0$ (we use that $0/\xi = 0$!). Therefore, a revision of the solution Heston by treating individual cases. We intend to do so in the future.

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