

On Stability of the Mechanical Lagrangian Systems

Valer Nîmineţ

Department of Mathematics and Informatics, Faculty of Sciences, "Vasile Alecsandri" University of Bacau, Romania.
valern@ub.ro

Otilia Lungu

Department of Mathematics and Informatics, Faculty of Sciences, "Vasile Alecsandri" University of Bacau. Romania.
otilia.lungu@ub.ro

Abstract. *We consider MLS (mechanical Lagrangian systems) with external forces. We give some conditions of stability and dissipativity and show that the energy of the system decreases on the integral curves.*

Key words: LMS, stability, dissipative system.

1. Introduction.

We consider a Lagrangian system

$$\Sigma = (M, L(x, y), F(x, y))$$

with M a differential manifold, L a regular Lagrangian and $F(x, y)$ external forces seen as d-covector field on TM .

The evolution equations of Σ are:

$$(1.1) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = F_i(x, y), \quad y^i = \frac{dx^i}{dt}.$$

Expanding the derivative with respect to t , replacing the derivatives $\frac{\partial^2 L}{\partial y^i \partial y^j}$ with $2g_{ij}$ and multiplying with (g^{jk}) , the equation (1.1) becomes system of differential equations of second order:

$$(1.2.) \quad \frac{d^2 x^i}{dt^2} + 2G^i(x, \dot{x}) = \frac{1}{2} F^i(x, \dot{x})$$

where

$$G^i(x, y) = \frac{1}{4} g^{ij} \left(\frac{\partial^2 L}{\partial y^j \partial x^k} y^k - \frac{\partial L}{\partial x^i} \right), y = \dot{x} = \frac{dx^i}{dt}$$

with the notation $y^i = \frac{dx^i}{dt}$, equations (1.2.) are equivalent with the system of equations:

$$(1.3.) \quad \begin{aligned} \frac{dx^i}{dt} &= y^i \\ \frac{dy^i}{dt} &= -2(G^i - \frac{1}{4} F^i) \end{aligned}$$

The solutions of this system can be seen as integrable curves of the vector field S^* on TM given by:

$$(1.4.) \quad \begin{aligned} S^* &= y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i} \\ G^{*i} &= G^i - \frac{1}{4} F^i \end{aligned}$$

and also is a semispray.

The semispray associated to Lagrangian L is

$$(1.4') \quad S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$$

$\tilde{S}$ is a vector field on the TM which depends only of the MLS.

2. Dissipative control of MLS.

Definition 2.1. The mechanical system Σ is **dissipative** if the force F is dissipative i.e. $F_j(x, y)y^j \leq 0$ and Σ is **strict dissipative** if the force F is strict dissipative i.e. $F_j(x, y)y^j \leq -\alpha y_j y^j$, with $\alpha > 0$ and $y_j = g_{ij}(x, y)y^i$.

The conditions of dissipativity and of strictly dissipativity can be also formulated as it follows:

If the matrix $(g_{ij}(x, y))$ is positively defined, it defines a Riemannian metric g in the vertical bundle over TM, by formulae

$$(2.1.) \quad g(A, B) = g_{ij} A^i B^j$$

for vertical fields $A = A^i \frac{\partial}{\partial y^i}$ and $B = B^j \frac{\partial}{\partial y^j}$.

The force field (F^i) can be seen as a section in vertical bundle by definition $F = F^i(x, y) \frac{\partial}{\partial y^i}$.

The Liouville field $C = y^i \frac{\partial}{\partial y^i}$ appears as a section in vertical bundle and we have

$$(2.2.) \quad g(C, C) = g_{ij} y^i y^j = y_i y^i := \|y\|^2$$

Also, we have $F_j(x, y) y^j = g(F, C)$ and so, the dissipativity condition one writes $g(F, C) \leq 0$ and the strictly dissipation condition becomes $g(F, C) \leq -\alpha \|y\|^2$, when the system is dissipative, the energy of the system decreases on the integral curves of the equations (1.3.):

Theorem 2.1. *If the Lagrangian system Σ is dissipative, then its energy $E(x, y) = y^i \frac{\partial L}{\partial y^i} - L$ decreases on the solutions curves of the equations (1.3).*

If the system Σ is strictly dissipative and the solutions curves have not singularities, then the energy E is strictly decreasing on solutions curves of the equations (1.3).

Proof. Let $\gamma: t \rightarrow (x(t), y(t)), y = \dot{x}$ a curve on TM, solution of the system $\nabla_{\dot{\gamma}} \dot{\gamma} = F \circ \gamma$, where we noted with ∇ Levi-Civita connexion of (M, g) manifold. Along this curve, we have:

$$\begin{aligned} \frac{dE}{dt} &= \ddot{x}^i \frac{\partial L}{\partial \dot{x}^i} + \dot{x}^i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) - \ddot{x}^i \frac{\partial L}{\partial \dot{x}^i} - \dot{x}^i \frac{\partial L}{\partial x^i} - \ddot{x}^i \frac{\partial L}{\partial \dot{x}^i} = \\ &= \dot{x}^i \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} \right) = \dot{x}^i F_i(x, \dot{x}) \leq 0 \end{aligned}$$

So E is decreasing on γ .

If Σ is strictly dissipative, then

$$\frac{dE}{dt} = \ddot{x}F_i(x, \dot{x}) \leq -\alpha \|\dot{x}\|^2 < 0$$

so E is strictly decreasing on γ .

We define the equilibrium point of the system Σ as zeros of the semispray S^* .

From (1.4.) it results that the equilibrium point of the system Σ are $(x_0^i, 0)$ or $O_{x_0} \in T_{x_0}M$, where (x_0^i) must be a solution of the equations

$$G^{*i}(x_0^i, 0) = 0 \Leftrightarrow G^i(x_0^i, 0) - \frac{1}{4}F^i(x_0^i, 0) = 0$$

For a Lagrange manifold (M, L) , the tangent manifold TM is a Riemannian manifold assuming that $g_{ij}(x, y) = \frac{\partial^2 L}{\partial y^i \partial y^j}$ is positively defined.

The Riemannian metric on TM is

$$g_L = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \delta y^j$$

with

$$\delta y^i = dy^i + N_j^i dx^j$$

where (N_j^i) are the coefficients of the nonlinear connection defined by the

semispray $N_j^i = \frac{\partial G^i}{\partial y^j}$.

If the manifold (TM, G) is complete as a metric space one can be given theorems of stability for the equilibrium points of a vector field on TM, similar to those from R^n .

To give sufficient condition so that the system will be stable, we use a Lyapunov function [2] constructed with the energy of the system.

So, we have:

Let $\Sigma = (M, L(x, y), F(x, y))$ be a dissipative Lagrangian systems with (TM, g_L) a complete Riemannian manifold. Let $(x_0^i, 0)$ be an equilibrium point of Σ , that is a zero of the vector field S^* . We suppose that $(x_0^i, 0)$ is a point of absolute minimum for the energy E of the system Σ . Then $(x_0^i, 0)$ is a stable equilibrium point.

And also:

Let $\Sigma = (M, L(x, y), F(x, y))$ be a dissipative Lagrangian system with (TM, g_L) a complete Riemannian manifold, $L \geq 0$ and L a homogenous function of degree $m \geq 2$ in the variables y . Let $(x_0^i, 0)$ be a point on TM with $F^i(x_0^i, 0) = 0$. Then, $(x_0^i, 0)$ is a stable point of S^* .

Conclusion.

The energy of the system decreases on the integral curves.

References

- [1].Miron, R., Anastasiei, M. (1994). *The Geometry of Lagrange Spaces. Theory and Applications*. Kluwer Academic Publishers FTPH no. 59.
- [2].Munoz-Lecanda, M.C., Yaniz-Fernandez, JKF. (2002) "Dissipative control of mechanical systems: a geometric approach", *SIAM Journal of Control Optimization*, 40(5), 2002, 1505-1516.
- [3].Nijmerjer H., Schaft van der A. (1990) *Nonlinear dynamical control systems*, Springer Verlag, 1990.
- [4].Nimineț V. (2006) "New results for mechanical lagrangian control systems", *Studii și Cercetări Științifice. Seria Matematică*, Nr. 16, 177-184.