Neutrosophic Hedge Algebras

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Abstract
We introduce now for the first time the neutrosophic hedge algebras as an extension of classical hedge algebras, together with an application of neutrosophic hedge algebras.

1. Introduction
The classical hedge algebras deal with linguistic variables. In neutrosophic environment we have introduced the neutrosophic linguistic variables. We have defined neutrosophic partial relationships between single-valued neutrosophic numbers. Neutrosophic operations are used in order to aggregate the neutrosophic linguistic values.

2. Materials and Methods
We introduce now, for the first time, the Neutrosophic Hedge Algebras, as extension of classical Hedge Algebras.

Let's consider a Linguistic Variable:
with \( \text{Dom}(x) \) as the word domain of \( x \), whose each element is a word (label), or string of words.

Let \( \mathcal{A} \) be an attribute that describes the value of each element \( x \in \text{Dom}(x) \), as follows:
\[
\mathcal{A}: \text{Dom}(x) \rightarrow [0, 1]^3
\]  
\( \mathcal{A}(x) \) is the neutrosophic value of \( x \) with respect to this attribute:
\[
A(x) = \langle t_x, i_x, f_x \rangle
\]  
where \( t_x, i_x, f_x \in [0, 1] \), such that

- \( t_x \) means the degree of value of \( x \);
- \( i_x \) means the indeterminate degree of value of \( x \);
- \( f_x \) means the degree of non-value of \( x \).

We may also use the notation: \( x\{t_x, i_x, f_x\} \).

A neutrosophic partial relationship \( \leq_N \) on \( \text{Dom}(x) \), defined as follows:
\[
x\{t_x, i_x, f_x\} \leq_N y\{t_y, i_y, f_y\},
\]  
if and only if \( t_x \leq t_y \), and \( i_x \geq i_y, f_x \geq f_y \).

Therefore, \( (\text{Dom}(x), \leq_N) \) becomes a neutrosophic partial order set (or neutrosophic poset), and \( \leq_N \) is called a neutrosophic inequality.

Let \( C = \{0, w, 1\} \) be a set of constants, \( C \subset \text{Dom}(x) \), where:

- \( 0 \) = the least element, or \( 0_{(0,1,1)} \);
- \( w \) = the neutral (middle) element, or \( w_{(0.5,0.5,0.5)} \);
- \( 1 \) = the greatest element, or \( 1_{(1,0,0)} \).

Let \( G \) be a word-set of two neutrosophic generators, \( G \subset \text{Dom}(x) \), qualitatively a negative primary neutrosophic term (denoted \( g^- \)), and the other one that is qualitatively a positive primary neutrosophic term (denoted \( g^+ \)), such that:
\[0 \leq g^- \leq w \leq g^+ \leq 1, \quad (4)\]

or transcribed using the neutrosophic components:

\[0 \leq t_{g^-} \leq 0.5 \leq t_{g^+} \leq 1\]

(here there are classical inequalities)

\[1 \geq i_{g^-} \geq 0.5 \geq i_{g^+} \geq 0, \quad \text{and} \]

\[1 \geq f_{g^-} \geq 0.5 \geq f_{g^+} \geq 0.\]

Let \(H \subset \text{Dom}(x)\) be the set of neutrosophic hedges, regarded as unary operations. Each hedge \(h \in H\) is a functor, or comparative particle for adjectives and adverbs as in the natural language (English).

\[h: \text{Dom}(x) \to \text{Dom}(x)\]

\[x \to h(x). \quad (5)\]

Instead of \(h(x)\) one easily writes \(hx\) to be closer to the natural language.

By associating the neutrosophic components, one has:

\[h_{\langle t_h, i_h, f_h \rangle} x_{\langle t_x, i_x, f_x \rangle}.\]

A hedge applied to \(x\) may increase, decrease, or approximate the neutrosophic value of the element \(x\).

There also exists a neutrosophic identity \(I \in \text{Dom}(x)\), denoted \(I_{\langle 0,0,0 \rangle}\) that does not hang on the elements:

\[I_{\langle 0,0,0 \rangle} x_{\langle t_x, i_x, f_x \rangle}.\]

In most cases, if a hedge increases / decreases the neutrosophic value of an element \(x\) situated above the neutral element \(w\), the same hedge does the opposite, decreases / increases the neutrosophic value of an element \(y\) situated below the neutral element \(w\).

And reciprocally.

If a hedge approximates the neutrosophic value, by diminishing it, of an element \(x\) situated above the neutral element \(w\), then it approximates the neutrosophic value, by enlarging it, of an element \(y\) situated below the neutral element \(w\).

Let's refer the hedges with respect to the upper part \((\cup)\), above the neutral element, since for the lower part \((L)\) it will automatically be the opposite effect.

We split the set of hedges into three disjoint subsets:

\[H_{\cup}^{+} = \text{the hedges that increase the neutrosophic value of the upper elements};\]

\[H_{\cup}^{-} = \text{the hedges that decrease the neutrosophic value of the upper elements};\]

\[H_{\cup}^{\sim} = \text{the hedges that approximate the neutrosophic value of the upper elements}.\]

**Notations:** Let \(x = x_{\cup} \cup w \cup x_{\downarrow}\), where \(x_{\downarrow}\) cons-titutes the upper element set, while \(x_{\downarrow}\) the lower element subset, \(w\) the neutral element. \(x_{\downarrow}\) and \(x_{\downarrow}\) are disjoint two by two.

### 3. Operations on Neutrosophic Components

Let \(\langle t_1, i_1, f_1 \rangle, \langle t_2, i_2, f_2 \rangle\) neutrosophic numbers.

Then:

\[t_1 + t_2 = \begin{cases} t_1 + t_2, & \text{if } t_1 + t_2 \leq 1; \\ 1, & \text{if } t_1 + t_2 > 1; \end{cases} \quad (6)\]

and
\[ t_1 - t_2 = \begin{cases} 0, & \text{if } t_1 - t_2 < 0; \\ t_1 - t_2, & \text{if } t_1 - t_2 \geq 0. \end{cases} \]  

(7)

Similarly for \( i_1 \) and \( f_1 \):

\[ i_1 + i_2 = \begin{cases} i_1 + i_2, & \text{if } i_1 + i_2 \leq 1; \\ 1, & \text{if } i_1 + i_2 > 1; \end{cases} \]  

(8)

\[ i_1 - i_2 = \begin{cases} 0, & \text{if } i_1 - i_2 < 0; \\ i_1 - i_2, & \text{if } i_1 - i_2 \geq 0. \end{cases} \]  

(9)

and

\[ f_1 + f_2 = \begin{cases} f_1 + f_2, & \text{if } f_1 + f_2 \leq 1; \\ 1, & \text{if } f_1 + f_2 > 1; \end{cases} \]  

(10)

\[ f_1 - f_2 = \begin{cases} 0, & \text{if } f_1 - f_2 < 0; \\ f_1 - f_2, & \text{if } f_1 - f_2 \geq 0. \end{cases} \]  

(11)

4. Neutrosophic Hedge-Element Operators

We define the following operators:

4.1. Neutrosophic Increment

Hedge \( \uparrow \) Element = \( \langle t_1, i_1, f_1 \rangle \uparrow \langle t_2, i_2, f_2 \rangle = \langle t_2 + t_1, i_2 - i_1, f_2 - f_1 \rangle \),

(12)

meaning that the first triplet increases the second.

4.2. Neutrosophic Decrement

Hedge \( \downarrow \) Element = \( \langle t_1, i_1, f_1 \rangle \downarrow \langle t_2, i_2, f_2 \rangle = \langle t_2 - t_1, i_2 + i_1, f_2 + f_1 \rangle \),

(13)

meaning that the first triplet decreases the second.

4.3. Theorem 1

The neutrosophic increment and decrement operators are non-commutative.

5. Neutrosophic Hedge-Hedge Operators

Hedge \( \uparrow \) Hedge = \( \langle t_1, i_1, f_1 \rangle \uparrow \langle t_2, i_2, f_2 \rangle = \langle t_1 + t_2, i_1 + i_2, f_1 + f_2 \rangle \)

(14)

Hedge \( \downarrow \) Hedge = \( \langle t_1, i_1, f_1 \rangle \downarrow \langle t_2, i_2, f_2 \rangle = \langle t_1 - t_2, i_1 - i_2, f_1 - f_2 \rangle \)

(15)

6. Neutrosophic Hedge Operators

Let \( x_\uparrow (x_\uparrow, i_\uparrow, f_\uparrow) \in Dom(x) \) i.e. \( x_\uparrow \) is an upper element of \( Dom(x) \), and

\( h_\uparrow (h_\uparrow, i_\uparrow, f_\uparrow) \in H_\uparrow \),

\( h_\downarrow (h_\downarrow, i_\downarrow, f_\downarrow) \in H_\downarrow \),

then \( h_\downarrow \) applied to \( x_\uparrow \) gives

\( (h_\uparrow x_\uparrow)(t, i_\uparrow, f_\uparrow) \uparrow \langle t, i_\uparrow, f_\uparrow \rangle \),

and \( h_\uparrow \) applied to \( x_\uparrow \) gives

\( (h_\downarrow x_\uparrow)(t, i_\uparrow, f_\uparrow) \downarrow \langle t, i_\uparrow, f_\uparrow \rangle \).

(16)

(17)

(18)
Now, let \( x_L(t_{x_L}, i_{x_L}, f_{x_L}) \in Dom(x_L) \), i.e. \( x_L \) is a lower element of \( Dom(x) \). Then, \( h^+_u \) applied to \( x_L \) gives:
\[
h^+_u x_L(t_{x_L}, i_{x_L}, f_{x_L}) \uplus (t_{h^+_u}, i_{h^+_u}, f_{h^+_u}),
\]
and \( h^-_u \) applied to \( x_L \) gives:
\[
h^-_u x_L(t_{x_L}, i_{x_L}, f_{x_L}) \uplus (t_{h^-_u}, i_{h^-_u}, f_{h^-_u}).
\]
In the same way, we may apply many increasing, decreasing, approximate or other type of hedges to the same upper or lower element
\[
h^+_u \circ h^+_{u-1} \circ h^+_{u-2} \circ \ldots \circ h^+_{u_1} x,
\]
generating new elements in \( Dom(x) \).
The hedges may be applied to the constants as well.

6.1. Theorem 2

A hedge applied to another hedge weakens or strengthens or approximates it.

6.2. Theorem 3

If \( h^+_u \in H^+ \) and \( x_L \in Dom(x_L) \), then \( h^+_u x_L \geq x_L \).
If \( h^-_u \in H^- \) and \( x_L \in Dom(x_L) \), then \( h^-_u x_L \geq x_L \).
If \( h^+_u \in H^+ \) and \( x_L \in Dom(x_L) \), then \( h^+_u x_L \leq_N x_L \).
If \( h^-_u \in H^- \) and \( x_L \in Dom(x_L) \), then \( h^-_u x_L \leq_N x_L \).

6.3. Converse Hedges

Two hedges \( h_1 \) and \( h_2 \in H \) are converse to each other, if \( \forall x \in Dom(x), h_1 x \leq_N x \) is equivalent to \( h_2 x \geq_N x \).

6.4. Compatible Hedges

Two hedges \( h_1 \) and \( h_2 \in H \) are compatible, if \( \forall x \in Dom(x), h_1 x \leq_N x \) is equivalent to \( h_2 x \leq_N x \).

6.5. Commutative Hedges

Two hedges \( h_1 \) and \( h_2 \in H \) are commutative, if \( \forall x \in Dom(x), h_1 h_2 x = h_2 h_1 x \). Otherwise they are called non-commutative.

6.6. Cumulative Hedges

If \( h^+_u \) and \( h^+_u \in H^+ \), then two neutrosophic edges can be cumulated into one:
\[
h^+_u (t_{h^+_u}, i_{h^+_u}, f_{h^+_u}) h^+_u (t_{h^+_u}, i_{h^+_u}, f_{h^+_u}) \uplus (t_{h^+_u}, i_{h^+_u}, f_{h^+_u}).
\]
Similarly, if \( h^-_u \) and \( h^-_u \in H^- \), then we can cumulate them into one:
\[
h^-_u (t_{h^-_u}, i_{h^-_u}, f_{h^-_u}) h^-_u (t_{h^-_u}, i_{h^-_u}, f_{h^-_u}) \uplus (t_{h^-_u}, i_{h^-_u}, f_{h^-_u}).
\]
Now, if the two hedges are converse, \( h^+_u \) and \( h^-_u \), but the neutrosophic components of the first (which is actually a neutrosophic number) are greater than the second, we cumulate them into one as follows:

\[
h^+_3 = (h^+_1, h^-_2) \langle t_{h^+_u}, i_{h^+_u}, f_{h^+_u} \rangle \bigcup (t_{h^-_u}, i_{h^-_u}, f_{h^-_u}) = (h^+_3, i^+_3, f^+_3).
\]  

But, if the neutrosophic components of the second are greater, and the hedges are commutative, we cumulate them into one as follows:

\[
h^+_3 = (h^+_1, h^-_2) \langle t_{h^-_u}, i_{h^-_u}, f_{h^-_u} \rangle \bigcup (t_{h^+_u}, i_{h^+_u}, f_{h^+_u}) = (h^+_3, i^+_3, f^+_3).
\]

7. Neutrosophic Hedge Algebra

\( NHA = (x, G, C, H \cup I, \leq_N) \) constitutes an abstract algebra, called Neutrosophic Hedge Algebra.

7.1. Example of a Neutrosophic Hedge Algebra \( \tau \)

Let \( G = \{Small, Big\} \) the set of generators, repres-ented as neutrosophic generators as follows:

\( Small(0.3,0.6,0.7), Big(0.7,0.2,0.3) \).

Let \( H = \{Very, Less\} \) the set of hedges, repres-ented as neutrosophic hedges as follows:

\( Very(0.1,0.1,0.1), Less(0.1,0.2,0.3) \),

where \( Very \in H^+_u \) and \( Less \in H^-_u \).

\( x \) is a neutrosophic linguistic variable whose domain is \( G \) at the beginning, but extended by generators.

The neutrosophic constants are:

\( C = \{0(0,1,1), Medium(0.5,0.5,0.5), 1(1,0,0)\} \).

The neutrosophic identity is \( I(0,0,0) \).

We use the neutrosophic inequality \( \leq_N \), and the neutrosophic increment / decrement operators previously defined.

Let's apply the neutrosophic hedges in order to generate new neutrosophic elements of the neutrosophic linguistic variable \( x \).

\( Very \) applied to \( Big \) [upper element] has a positive effect:

\( Very(0.1,0.1,0.1)Big(0.7,0.2,0.3) = (Very Big)(0.7+0.1,0.2+0.1,0.3-0.1) = (Very Big)(0.8,0.1,0.2) \).

Then:

\( Very(0.1,0.1,0.1)(Very Big)(0.9,0.1,0.2) = (Very Very Big)(0.9,0.0,0.1) \).

\( Very \) applied to \( Small \) [lower element] has a negative effect:

\( Very(0.1,0.1,0.1)Small(0.3,0.6,0.7) = (Very Small)(0.3-0.1,0.6+0.1,0.7+0.1) = (Very Small)(0.2,0.7,0.8) \).

If we compute \( (Very Very) \) first, which is a neutrosophic hedge-hedge operator:

\( (Very Very)(0.2,0.2,0.2)\),

and we apply it to \( Big \), we get:

\( (Very Very)(0.2,0.2,0.2)Big(0.7,0.2,0.3) = (Very Very Big)(0.7+0.2,0.2-0.2,0.3-0.2) = (Very Very Big)(0.9,0.0,0.1) \),

so, we get the same result.

\( Less \) applied to \( Big \) has a negative effect:
Less\((0.1,0.2,0.3)\) Big\((0.7,0.2,0.3)\) = \((\text{Less Big})_{(0.7-0.1,0.2+0.2,0.3)}\) = \((\text{Less Big})_{(0.6,0.4,0.6)}\).

Less applied to Small has a positive effect:

\[ \text{Less}_{(0.1,0.2,0.3)} \text{Small}_{(0.3,0.6,0.7)} = (\text{Less Small})_{(0.1+0.3,0.6-0.2,0.7-0.3)} = (\text{Less Small})_{(0.4,0.4,0.4)}. \]

The set of neutrosophic hedges \(H\) is enriched through the generation of new neutrosophic hedges by combining a hedge with another one using the neutrosophic hedge-hedge operators.

Further, the newly generated neutrosophic hedges are applied to the elements of the linguistic variable, and more new elements are generated.

Let's compute more neutrosophic elements:

\[ VLB = \text{Very}_{(0.1,0.1,0.1)} \text{Less}_{(0.1,0.2,0.3)} \text{Big}_{(0.7,0.2,0.3)} = (\text{Very Less Big}) \left[ (0.1,0.1,0.1) \left\lceil \frac{1}{h} (0.1,0.2,0.3) \right\rceil (0.7,0.2,0.3) \right] \]

\[ = (\text{Very Less Big})_{(0.1+0.1,0.1+0.2,0.1+0.3)} (0.7,0.2,0.3) \]

\[ = (\text{Very Less Big})_{(0.7-0.2,0.2-0.3,0.3-0.4)} = (\text{Very Less Big})_{(0.5,0.0)} \]

\[ VM = \text{Very}_{(0.1,0.1,0.1)} \text{Medium}_{(0.5,0.5,0.5)} = (\text{Very Medium})_{(0.1,0.1,0.1)} (0.5,0.5,0.5) \]

\[ = (\text{Very Medium})_{(0.6,0.4,0.4)} \]

\[ LM = \text{Less}_{(0.1,0.2,0.3)} \text{Medium}_{(0.5,0.5,0.5)} = (\text{Less Medium})_{(0.1,0.2,0.3)} (0.5,0.5,0.5) \]

\[ = (\text{Less Medium})_{(0.4,0.7,0.8)} \]

\[ VVS = \text{Very}_{(0.1,0.1,0.1)} \text{Very}_{(0.1,0.1,0.1)} \text{Small}_{(0.3,0.6,0.7)} = (\text{Very Very})_{(0.2,0.2,0.2)} \text{Small}_{(0.3,0.6,0.7)} = (\text{Very Small})_{(0.1,0.8,0.9)} \]

\[ VLS = \text{Very}_{(0.1,0.1,0.1)} \text{Less}_{(0.1,0.2,0.3)} \text{Small}_{(0.3,0.6,0.7)} = \text{Very}_{(0.1,0.1,0.1)} (\text{Less Small})_{(0.4,0.4,0.4)} \]

\[ = (\text{Very Less Small})_{(0.5,0.3,0.3)} \]

\[ \text{LAMax} = \text{Less}_{(0.1,0.2,0.3)} \text{Absolute Maximum}_{(1,0,0)} = (\text{Less Absolute Maximum})_{(0.1,0.2,0.3)} (1,0,0) \]

\[ = (\text{Less Absolute Maximum})_{(0.9,0.2,0.3)} \]

\[ \text{LAMin} = \text{Less}_{(0.1,0.2,0.3)} \text{Absolute Minimum}_{(0,1,1)} = (\text{Less Absolute Minimum})_{(0.1,0.2,0.3)} (1,0,1) \]

\[ = (\text{Less Absolute Maximum})_{(0.1,0.8,0.7)} \]

### 7.2. Theorem 4

Any increasing hedge \(h_{(t,i,f)}\) applied to the absolute maximum cannot overpass the absolute maximum.

**Proof:**

\[ h_{(t,i,f)} \left\lceil 1_{(1,0,0)} \right\rceil (h1)_{(1+t,0-0-f)} = (h1)_{(1,0,0)} = 1_{(1,0,0)}. \]

### 7.3. Theorem 5

Any decreasing hedge \(h_{(t,i,f)}\) applied to the absolute minimum cannot pass below the absolute minimum.

**Proof:**

\[ h_{(t,i,f)} \left\lfloor 0_{(0,1,1)} \right\rfloor (h0)_{(0-t,1+t,1+f)} = (h0)_{(0,1,1)} = 0_{(0,1,1)}. \]

### 8. Diagram of the Neutrosophic Hedge Algebra \(\tau\)

\[ 1_{(1,0,0)} \quad \text{ABSOLUTE MAXIMUM} \]

\[ VVB_{(0.9,0,0.1)} \quad \text{Very Very Big} \]
In this paper, the classical hedge algebras have been extended for the first time to
neutrosophic hedge algebras. With respect to an attribute, we have inserted the neutrosophic
degrees of membership / indeterminacy / nonmembership of each generator, hedge, and constant.
More than in the classical hedge algebras, we have introduced several numerical hedge operators:
for hedge applied to element, and for hedge combined with hedge. An extensive example of a
neutrosophic hedge algebra is given, and important properties related to it are presented.

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