BRAIN. Broad Research in Artificial Intelligence and Neuroscience,
ISSN 2067-3957, Volume 1, July 2010,
Special Issue on Complexity in Sciences and Artificial Intelligence,
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OPTIMIZATION PROBLEMS ON THRESHOLD GRAPHS

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Abstract. During the last three decades, different types of decompositions have been processed in the field of graph theory. Among these we mention: decompositions based on the additivity of some characteristics of the graph, decompositions where the adjacency law between the subsets of the partition is known, decompositions where the subgraph induced by every subset of the partition must have predeterminate properties, as well as combinations of such decompositions.

In this paper we characterize threshold graphs using the weakly decomposition, determine: density and stability number, Wiener index and Wiener polynomial for threshold graphs.

Keywords: *Threshold graph, weakly decomposition, Wiener index, Wiener polynomial.*

2000 *Mathematics Subject Classification*: 05C99, 68R10.

1.Preliminary results

1.1. Weakly decomposition

At first, we recall the notions of weakly component and weakly decomposition.

Definition 1. ([5], [18], [19]) *A set $A \subset V(G)$ is called a weakly set of the graph G if $N_G(A) \neq V(G) - A$ and $G(A)$ is connected. If A is a weakly set, maximal with respect to set inclusion, then $G(A)$ is called a weakly component. For simplicity, the weakly component $G(A)$ will be denoted with A. If A is a weakly set, then the partition $\{A, N(A), V - A \cup N(A)\}$ is called a weakly decomposition of G with respect to A.*

Below we remind a characterization of the weakly decomposition of a graph.

The name of "*weakly component*" is justified by the following result.

Theorem 1. ([6], [18], [19]) *Every connected and non-complete graph $G = (V, E)$ admits a weakly component A such that $G(V - A) = G(N(A)) + G(\overline{N}(A))$.*

Let $A \subset V$. Then A is a weakly component of G if and only if $G(A)$ is connected and $N(A) \sim \overline{N}(A)$.

The next result, that follows from Theorem 1, ensures the existence of a weakly decomposition in a connected and non-complete graph.

Corollary 1. *If $G = (V, E)$ is a connected and non-complete graph, then V admits a weakly decomposition (A, B, C), such that $G(A)$ is a weakly component and $G(V - A) = G(B) + G(C)$.*

Theorem 1 provides an $O(n + m)$ algorithm for building a weakly decomposition for a non-complete and connected graph.

Algorithm for the weakly decomposition of a graph ([18])

Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.
Output: A partition $V = (A, N, R)$ such that $G(A)$ is connected, $N = N(A)$, $A \not\sim R = \overline{N}(A)$.

begin

$A :=$ any set of vertices such that
$A \cup N(A) \neq V$
$N := N(A)$
$R := V - A \cup N(A)$
while ($\exists n \in N$, $\exists r \in R$ such that $nr \notin E$) *do*
begin
$A := A \cup \{n\}$

$N := (N - \{n\}) \cup (N(n) \cap R)$
$R := R - (N(n) \cap R)$
end
end

1.2. Threshold graphs

In this subsection we remind some results on threshold graphs.

A graph G is called *threshold* graph if $N_G(x) \subseteq N_G[y]$ or $N_G(y) \subseteq N_G[x]$ for any pair of vertices x and y in G.

Threshold graphs were first introduced by Chvátal and Hammer ([3]).

In [16], Ortiz and Villanueva-Ilufi give a structural characterization of threshold graphs for solving the following two difficult problems: enumeration of all maximal independent sets and the chromatic index problem.

Theorem 2. ([3]) A graph G is a threshold graph if and only if G does not contain a C_4, $\overline{C}_4$, P_4 as an induced subgraph.

Chvátal and Hammer also showed that threshold graphs can be recognizing in $O(n^2)$ time.

In [1], Babel showed that if G is a threshold graph then the algorithms that determine $\omega(G)$, $\chi(G)$, $\alpha(G)$ and $\theta(G)$ are $O(n+m)$ time.

Theorem 3. ([3]) A graph G is a threshold graph if and only if G is a cograph and G is a split graph.

In [4] (as well as in [10] and [14]) linear algorithms for recognizing a cograph can be found. Hammer and Simeone [11]) give an $O(n+m)$ algorithm for recognizing a split graph. Therefore, an algorithm that recognizes a threshold graph is $O(n(n+m))$.

In [15] a linear algorithm for recognizing a threshold graph can be found.

2.New results on threshold graphs

2.1. Characterization of a threshold graph using the weakly decomposition

In this paragraph we give a new characterization of threshold graphs using the weakly decomposition. Also, we determine the stability number and the clique number for threshold graphs.

Theorem 4. *Let G=(V,E) be a connected graph with at least two nonadjacent vertices and (A,N,R) a weakly decomposition, with A the weakly component. G is a threshold graph if and only if:*
i) $A \sim N \sim R$*;*

ii) $d_G(n) = |V| - 1$, $d_G(r) = |N|$, $\forall n \in N$, $\forall r \in R$;
iii) G(A) is threshold graph.

The above results lead to a recognition algorithm with the total execution time $O(n(n+m))$.

2.2. Determination of clique number and stability number for a threshold graph

The threshold graphs is a graph class of bounded clique-width ([2]).

Theorem 5. *Let G=(V,E) be connected with at least two non-adjacent vertices and (A,N,R) a weakly decomposition with A the weakly component. If G is a threshold graph then*

$$\alpha(G) = \alpha(G(A)) + |R| \text{ and } \omega(G) = \omega(G(A)) + |N|.$$

As a consequence of the above theorem, we give an algorithm that leads to a stable set of maximal cardinal and to a clique of maximal cardinal in a threshold graph.

Input: A threshold, connected graph with at least two nonadjacent vertices, $G = (V, E)$

Output: Determination of $\alpha(G)$ and $\omega(G)$

begin
  $S = \emptyset$; $Q = \emptyset$; $s := 0$; $q := 0$; $i := 1$; $G_i := G$;
  while $|V(G_i)| \geq 4$ *do*
    Determine a weakly decomposition (A_i, N_i, R_i) of G_i, with R_i stable, N_i clique and $G(A_i)$ threshold
    if (G_i is complete) *then*
      $S := S \cup \{v\}$, $s := s + 1$, $\forall v \in V(G_i)$;
      $Q := Q \cup V(G_i)$, $q := q + |V(G_i)|$
    else
      $S := S \cup R_i$, $s := s + |R_i|$;
      $Q := Q \cup N_i$, $q := q + |N_i|$;
      $i := i + 1$;
      $H := G_i$;
    $\alpha(G) := s + \alpha(H)$;
    $\omega(G) := q + \omega(H)$
end

Remark 1. The most time consuming operation inside the *while* loop is the determination of the decomposition (A, N, R), namely $O(n + m)$. As the *while* body executes at most n times, it follows that the total execution time is $O(n(n+m))$.

The characterization theorem of threshold graphs leads to the following result that is useful in the next section.

Corollary 2. *Let G=(V,E) be connected with at least two non-adjacent vertices and (A,N,R) a weakly decomposition with A the weakly component. If G is a threshold graph then if after k steps in the weakly decomposition algorithm of G we get* $|A_k| \leq 3$ *then* $A_k \simeq K_3$ *or* $A_k \simeq K_2$ *or* $A_k \simeq K_1$.

3.Some Applications in Optimization Problems

In this section we point some applications of threshold graphs in optimization problems.

Facility location analysis deals with the problem of finding optimal locations for one or more facilities in a given environment [13]. Location problems are classical optimization problems with many applications in industry and economy. The spatial location of the facilities often takes place in the context of a given transportation, communication, or transmission system. A first paradigme for location is based on the minimization of transportation cost.

According to their objective function, we can consider two types of location problems. The first type consists of those problems that use a minimax criterion. For example, if we want to determine the location of a hospital the main objective is to find a site that minimizes the maximum response time between the hospital and site of a possible emergency. More generally, the aim of the first problem type is to determine a location that minimizes the maximum distance to any other location in the network. The second type of location problems optimizes a "minimum of a sum" criterion, which is used in determining the location for a service facility like a shopping mall, for which we try to minimize the total travel time. The following centrality indices are defined in [13].

The eccentricity of a vertex u is $e_G(u) = max\{d(u,v)|v \in V\}$.

The radius is $r(G) = min\{e_G(u)|u \in V\}$.

The center of a graph G is $\mathcal{C}(G) = \{u \in V|r(G) = e_G(u)\}$.

We consider the second type of location problems. Suppose we want to place a service facility such that the total distance to all customers in the region is minimal. The problem of finding an appropriate location can be solved by computing the set of vertices with minimum total distance.

We denote the sum of the distances from a vertex u to any other vertex in a graph G=(V,E) as the total distance $s(u) = \sum_{v\in V} d(u,v)$. If the minimum total distance of G is denoted by $s(G) = min\{s(u)|u \in V\}$, the median $\mathcal{M}(G)$ of G is given by $\mathcal{M}(G) = \{u \in V|s(G) = s(u)\}$.

The Wiener index was introduced in 1947 by Horold Wiener ([20]) and is defined as the sum of distance between all pairs of vertices in G:

$$W(G) = \sum_{u,v\in V} d_G(u,v).$$

We wish to point out that the theoretical framework is especially well elaborated for the Wiener index of trees ([7]).

The distance-counting polynomial was introduced [12] as:

$$H(G,x) = \sum_k d(G,k)x^k,$$

with $d(G,0) = |V(G)|$ and $d(G,1) = |E(G)|$, where $d(G,k)$ is the number of pair vertices lying at distance k to each other. This polynomial was called Wiener, by its author Hosoya, in the more recent literature [9], [17].

Our result concerning the center of a threshold graph is the following.

Theorem 6. *Let G=(V,E) be a connected graph with at least two nonadjacent vertices. If G is threshold and if after k steps in the algorithm weakly decomposition of G we get* $|A_k| \leq 3$ *, then the center and the median are equal to* N*, the radius is 1, while the excentricity is 1 for the vertices in* N *and 2 for the others. Also*

$$H(G,x) = [\tfrac{1}{2}(\alpha(G)-1)^2 + |A_k|(\alpha(G)-1)]x^2 + |E(G)|x + |V(G)| \text{ and}$$
$$W(G) = |E(G)| + (\alpha(G)-1)^2 + 2|A_k|(\alpha(G)-1).$$

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